Journal of Global Optimization (2005) 32: 83–91
DOI 10.1007/s10898-004-5905-3

Integration of ε-Fenchel Subdifferentials and Maximal Cyclic Monotonicity

SYLVIE MARCELLIN and LIONEL THIBAULT
Département de Mathématiques, Université Montpellier II, CC 051, Place Eugéne Bataillon, 34095 Montpellier cedex 5, France (e-mail: {marcelin, thibault}@math.univ-montp2.fr)

(Received 29 July 2003; accepted in revised form 12 April 2004)

Abstract. This paper concerns the integration of ε-Fenchel subdifferentials of proper lower semicontinuous convex functions defined on arbitrary topological vector spaces. We make use of integration tools to provide a representation formula of the approximate subdifferential of convex functions, and also to identify the class of maximal cyclically monotone families of operators.

Mathematics Subject Classification: 46N10, 26B25, 49J52, 58C20, 52A41

Key words: Convex functions, Cyclically monotone families of operators, ε-Fenchel subdifferentials, Integration of ε-subdifferentials

0. Introduction and Preliminaries

The approximate or ε-Fenchel subdifferential of a function f from a real Hausdorff locally convex topological vector space X into $\mathbb{R} \cup \{+\infty\}$ is the set-valued mapping denoted by $\partial_\varepsilon f(\cdot) : X \to 2^{X^*}$ whose graph is given, for each real number $\varepsilon \geqslant 0$, by

$$\partial_\varepsilon f : = \{(x, x^*) \in X \times X^* : \langle x^*, u - x \rangle + f(x) - \varepsilon \leqslant f(u), \forall u \in X\}$$

where, as usual, X^* stands for the topological dual space of X. In the special case when f is convex and $\varepsilon = 0$, one recognizes the (Fenchel) subdifferential of convex analysis.

Note the elementary fact that, given any $\varepsilon > 0$, $\partial_\varepsilon f(x) \neq \emptyset$ whenever the convex function f is lower semicontinuous (lsc) on X and the point x belongs to its effective domain *dom* f: $= \{u \in X : f(u) < +\infty\}$. Recall that an extended-real-valued function is said to be proper if it does not take the value $-\infty$ and its effective domain is a nonempty set. Throughout the paper, the class of all proper lower semicontinuous convex functions on X will be denoted by $\Gamma_0(X)$.

The operator ∂_ε is of great importance in convex and variational analysis as well as in numerical optimization. It has been widely studied since its introduction in the work of Brøndsted and Rockafellar [1], and it benefits from a rich calculus as shown in Hiriart-Urruty [3], (see also [4, 5]).

In [2], Combari et al. established the first integration result for ε-Fenchel subdifferentials, showing that

$$(\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x),\ \forall \varepsilon > 0,\ \forall x \in X) \Longleftrightarrow f = g + \text{constant}$$

provided that the extended-real-valued function g is proper and f is proper convex and lower semicontinuous with respect to the weak topology $\sigma(X, X^*)$ of X.

In Section 1 of the present paper, we obtain another integration result for the approximate subdifferential of proper lower semicontinuous convex functions and we make use of it, to give a representation formula for $\partial_{\varepsilon+\delta}$ when $\varepsilon \geqslant 0, \delta > 0$ in terms of ∂_η with $0 \leqslant \eta \leqslant \delta$, that generalizes to locally convex topological spaces, the one obtained by Martinez-Legaz and Théra [6], for $\varepsilon \geqslant 0$ and $\delta = 0$ in the Banach setting.

In Section 2, we show how the integration result enables us to identify the class of maximal cyclically monotone families of operators in the sense of Verona and Verona (see [10]).

1. Integration of ε-Fenchel Subdifferentials

As mentioned in the introduction, in [2], one can find the first result of the type "$(\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \forall \varepsilon > 0, \forall x \in X) \Leftrightarrow (f = g + \text{constant})$" in the general framework of spaces paired in duality (see [7]), and where the function f belongs to $\Gamma_0(X)$ and g is only assumed to be proper. The proof of this early equivalence relies on a transitivity property of the approximate Fenchel subdifferential pointed out in Lemma 1.1 of [2]. Here, we will use the subdifferential calculus in terms of ε-subdifferentials initiated by Hiriart-Urruty and Phelps in [4] (see also [3, 5]), and precisely, the general composition rule of the following theorem, to provide an alternative proof of the integration result for ε- subdifferentials of proper lsc convex functions.

THEOREM 1.1 (*Hiriart-Urruty and Phelps* [4], *Th.* 3.1). *Suppose that E and F are real locally convex Hausdorff topological vector spaces. Let $A : E \to F$ be a continuous affine mapping with linear part denoted by A_0 and $f : F \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. Then, for any $x \in E$ such that $Ax \in \operatorname{dom} f$,*

$$\partial(f \circ A)(x) = \bigcap_{\varepsilon > 0} cl^*(A_0^* \partial_\varepsilon f(Ax)),$$

where A_0^* is the adjoint mapping of A_0 and cl^* means the closure operation with respect to the weak-star topology on the topological dual space of E.

Now, let us state and prove the main result of this section:

THEOREM 1.2. *Let X be a real locally convex Hausdorff topological vector space paired in duality with its topological dual space X^*. Let $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lsc convex functions and α be a positive real number. Then, the following are equivalent*:

(a) $\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x)$, $\forall x \in X$, $\forall \varepsilon \in \mathbb{R}$ *such that* $0 < \varepsilon < \alpha$,
(b) *there exists some constant real number C independent of α, such that $f = g + C$ on X.*

REMARK. In this theorem, we still work in the setting of spaces in duality (as in [2]) but here, the inclusion (a) is required only for positive ε below some arbitrary threshold, which is less restrictive. Note also that in Theorem 1.2, both functions lie in $\Gamma_0(X)$ (which is natural since the lower semicontinuity will hold *a posteriori* by virtue of the equality of the functions up to an additive constant).

Proof. The implication (b) $\Rightarrow$ (a) is obvious, so we will just establish the opposite one. The approach below is mainly inspired by the one adopted by Thibault to prove Proposition 3 in [9]. Let us assume (a). The function f is supposed to be proper and in view of (a), one has $\emptyset \neq \operatorname{dom} f \subset \operatorname{dom} g$. So without loss of generality, suppose that $0 \in \operatorname{dom} f$. Then $f(0) \in \mathbb{R}$ and $g(0) \in \mathbb{R}$. Fix any $b \in X$, denote $\Delta_b := \mathbb{R}b$ and define $I_b : \Delta_b \to X$ by $I_b(u) := u$ for all $u \in \Delta_b$. The mapping I_b is linear and continuous on Δ_b with respect to the topology induced by the one of X. Note that the functions $f \circ I_b$ and $g \circ I_b$ are convex lsc and proper (since $0_X \in \operatorname{dom} f \cap \Delta_b \subset \operatorname{dom} g \cap \Delta_b$) on Δ_b. Hence, Δ_b being one dimensional, $\operatorname{dom} \partial(f \circ I_b)$ is dense in $\operatorname{dom}(f \circ I_b)$. We prove the following:

$$\forall u \in \Delta_b, \quad \partial(f \circ I_b)(u) \subset \partial(g \circ I_b)(u). \tag{1.1}$$

Fix $u \in \Delta_b$. If $\partial(f \circ I_b)(u) = \emptyset$ then the inclusion (1.1) is obvious. So, suppose that $u \in \operatorname{dom} \partial(f \circ I_b)$ and consider any $u^* \in \partial(f \circ I_b)(u)$. According to Theorem 1.1, one has

$$u^* \in \bigcap_{\varepsilon > 0} \operatorname{cl}(I_b^* \partial_\varepsilon f(I_b u)),$$

where cl denotes just the closure, since Δ_b is finite dimensional. In particular,

$$u^* \in \bigcap_{0 < \varepsilon < \alpha} \operatorname{cl}(I_b^* \partial_\varepsilon f(I_b u)).$$

By virtue of (a), for all real number ε satisfying $0 < \varepsilon < \alpha$, one may write $\partial_\varepsilon f(I_b u) \subset \partial_\varepsilon g(I_b u)$ and hence

$$u^* \in \bigcap_{0<\varepsilon<\alpha} \mathrm{cl}(I_b^* \partial_\varepsilon g(I_b u)).$$

Given any $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$, observing that

$$0 < \varepsilon_1 \leqslant \varepsilon_2 \Rightarrow \partial_{\varepsilon_1} g(I_b u) \subset \partial_{\varepsilon_2} g(I_b u),$$

one actually has

$$u^* \in \bigcap_{\varepsilon>0} \mathrm{cl}(I_b^* \partial_\varepsilon g(I_b u)). \tag{1.2}$$

Applying Theorem 1.1 once again, the conclusion (1.2) ensures that $u^* \in \partial(g \circ I_b)(u)$. The element u^* in $\partial(f \circ I_b)(u)$ being arbitrary, one gets the expected inclusion in (1.1). Therefore, since Δ_b is one dimensional, the use of the left or right derivative (see [8], p. 239) entails that

$$f(b) = g(b) + f(0) - g(0). \tag{1.3}$$

The equality (1.3) holds for each vector $b \in X$. Consequently, $\mathrm{dom}\, f = \mathrm{dom}\, g$. Thus, putting $C := f(0) - g(0)$, we obtain a constant real number, independent of α such that $f(x) = g(x) + C$ for all $x \in X$, which completes the proof. □

We have already noticed that, given any proper convex function f, any point $x \in \mathrm{dom}\, f$ and any real numbers ε_1, ε_2 with $0 \leqslant \varepsilon_1 \leqslant \varepsilon_2$, one has $\partial_{\varepsilon_1} f(x) \subset \partial_{\varepsilon_2} f(x)$. Besides, it is not difficult to see that for each $v \geqslant 0$:

$$\partial_v f(x) = \bigcap_{\varepsilon>0} \partial_{v+\varepsilon} f(x),$$

which can be viewed as an "approximation from above" of $\partial_v f(x)$.

Adapting ideas of Martinez-Legaz and Théra ([6], Theorem 1) and using Theorem 1.2, we get some "estimation from below" of the graph of the approximate subdifferential, that is, a representation formula for the $(\varepsilon + \delta)$-subdifferential of a proper $\mathrm{I_{sc}}$ convex function in terms of its η-subdifferentials for all positive real numbers η not greater than $\delta(> 0)$, in the general context of locally convex topological spaces.

THEOREM 1.3. *Let X be a real locally convex Hausdorff topological vector space, $f \in \Gamma_0(X)$ and let $\varepsilon \geqslant 0$, $\delta > 0$ be real numbers. Then*

$$\begin{aligned}\partial_{\varepsilon+\delta}f = \{(x,x^*)\in X\times X^* : \langle x^*, x-x_0\rangle + \textstyle\sum_{i=0}^{m-1}\langle x_i^*, x_i - x_{i+1}\rangle \\ + \langle x_m^*, x_m - x\rangle + \textstyle\sum_{i=0}^{m}\delta_i \geqslant -\varepsilon-\delta, \\ \forall (x_i,x_i^*)\in\partial_{\delta_i}f, 0\leqslant\delta_i\leqslant\delta, i=0,1,\ldots,m,\ m\in\mathbb{N}\}.\end{aligned} \tag{1.4}$$

Proof. The inclusion "$\subset$" in (1.4) is immediate. Indeed, fix any $(\bar{x},x^*)$ in $\partial_{\varepsilon+\delta}f$, $m\in\mathbb{N}$, $(x_i,x_i^*)\in\partial_{\delta_i}f$ with $0\leqslant\delta_i\leqslant\delta$, $i=0,1,\ldots,m$. Then, one has

$$+\infty > f(x_0) \geqslant f(\bar{x}) + \langle x^*, x_0-\bar{x}\rangle - \varepsilon - \delta,$$

$$+\infty > f(x_{i+1}) \geqslant f(x_i) + \langle x_i^*, x_{i+1}-x_i\rangle - \delta_i, \quad 0\leqslant i\leqslant m-1,$$

$$+\infty > f(\bar{x}) \geqslant f(x_m) + \langle x_m^*, \bar{x}-x_m\rangle - \delta_m.$$

Summing up these inequalities, one obtains

$$-\varepsilon-\delta \leqslant \langle x^*, \bar{x}-x_0\rangle + \sum_{i=0}^{m-1}\langle x_i^*, x_i-x_{i+1}\rangle + \langle x_m^*, x_m-\bar{x}\rangle + \sum_{i=0}^{m}\delta_i.$$

Now, let us show the reverse inclusion. Denote by G the right hand side of (1.4). Because of the above inclusion and the fact that f belongs to $\Gamma_0(X)$, one has $G\neq\emptyset$. So, choose $(\bar{x},x^*)\in G$. Fix an arbitrary real number δ_0 with $0<\delta_0\leqslant\delta$, and any point $x_0\in\operatorname{dom} f$. Then take $x_o^*\in\partial_{\delta_0}f(x_0)$, which is possible since $f\in\Gamma_0(X)$. Hence, for all $y\in X$, define

$$\begin{aligned} g(y) := \sup\Bigg\{ & \sum_{i=0}^{m-1}\langle x_i^*, x_{i+1}-x_i\rangle + \langle x_m^*, y-x_m\rangle \\ & - \sum_{i=0}^{m}\delta_i : 0\leqslant\delta_i\leqslant\delta,\ (x_i,x_i^*)\in\partial_{\delta_i}f,\ i=1,\ldots,m,\ m\in\mathbb{N}\Bigg\}.\end{aligned}$$

Clearly, $g(X)\subset\mathbb{R}\cup\{+\infty\}$ and the function g is convex and lsc on X. Further, it is not difficult to see that

$$f(y)-f(x_0)\geqslant g(y) \quad \text{for all } y\in X.$$

As a consequence, $\operatorname{dom} f\subset\operatorname{dom} g$ and g is proper. Moreover, for $m=1$, $x_1^*:=x_0^*$, $x_1:=x_0$ and $\delta_1:=\delta_0$, according to the definition of $g(x_0)$, one has

$$-2\delta_0\leqslant g(x_0)(<+\infty). \tag{1.5}$$

We establish that, for all real number η with $0<\eta\leqslant\delta$ and all $z\in X$, we have $\partial_\eta f(z)\subset\partial_\eta g(z)$. First, note that whenever $z\notin\operatorname{dom} f$, one has $\partial_\eta f(z)=\varnothing$ for all positive η, and hence there is nothing to prove. So, fix η such that $0<\eta\leqslant\delta$, $z\in\operatorname{dom} f$ and $z^*\in\partial_\eta f(z)$. For all $y\in X$, by definition of $g(y)$, (putting $x_{m+1} := z$, $x^*_{m+1} := z^*$ and $\delta_{m+1} := \eta$ for any integer m) one may write

$$g(y) \geqslant g(z) + \langle z^*, y - z\rangle - \eta,$$

that is, $z^* \in \partial_\eta g(z)$. Thus,

$$\forall z \in X, \forall \eta \text{ such that } 0 < \eta \leqslant \delta,\ \partial_\eta f(z) \subset \partial_\eta g(z).$$

Then, we deduce from Theorem 1.2 that the functions f and g are equal up to an additive constant. In other words, for all $z \in X$

$$f(z) = g(z) + f(x_0) - g(x_0).$$

In particular,

$$f(x_0) - f(\bar{x}) - g(x_0) = -g(\bar{x}). \tag{1.6}$$

Further, for any $m \in \mathbb{N}$, δ_i with $0 \leqslant \delta_i \leqslant \delta$, and $(x_i, x_i^*) \in \partial_{\delta_i} f$, $i = 1, \ldots, m$, using the inclusion $(\bar{x}, x^*) \in G$ and the definition of G, we see that

$$\sum_{i=0}^{m-1} \langle x_i^*, x_{i+1} - x_i\rangle + \langle x_m^*, \bar{x} - x_m\rangle - \sum_{i=0}^{m} \delta_i \leqslant \langle x^*, \bar{x} - x_0\rangle + \varepsilon + \delta,$$

which leads to the inequality $g(\bar{x}) \leqslant \langle x^*, \bar{x} - x_0\rangle + \varepsilon + \delta$. Necessarily, by (1.6) one has:

$$f(x_0) - f(\bar{x}) - g(x_0) \geqslant \langle x^*, x_0 - \bar{x}\rangle - \varepsilon - \delta > -\infty,$$

and $\bar{x} \in \operatorname{dom} f$. Thus,

$$f(x_0) - f(\bar{x}) \geqslant \langle x^*, x_0 - \bar{x}\rangle - \varepsilon - \delta + g(x_0) \geqslant \langle x^*, x_0 - \bar{x}\rangle - \varepsilon - \delta - 2\delta_0,$$

where the last inequality is due to (1.5). Making $\delta_0 \downarrow 0$, we get

$$f(x_0) - f(\bar{x}) \geqslant \langle x^*, x_0 - \bar{x}\rangle - \varepsilon - \delta,$$

for all $x_0 \in \operatorname{dom} f$ and hence for all $x_0 \in X$. This exactly means that $x^* \in \partial_{\varepsilon+\delta} f(\bar{x})$. The second inclusion holds and the proof is complete. □

2. Cyclically Monotone Families of Operators and Maximality

In this section, the integration result of Theorem 1.2 will be the main tool in identifying the class of maximal cyclically monotone families of operators. The concept of cyclically monotone family of operators was first introduced by Verona-Verona [10]. Fix a line segment A in $\mathbb{R}$. Given a real Hausdorff topological vector space X with topological dual space X^* (paired in duality by $\langle ., .\rangle$), a family $\{T_\alpha : X \to 2^{X^*}, \alpha \in A\}$ of operators is said to be cyclically monotone if for all integer $m \geqslant 1$, $\alpha_i \in A$, $(x_i, x_i^*) \in X \times X^*$ with $x_i^* \in T_{\alpha_i}(x_i)$, $i = 0, 1, \ldots, m$, one has

$$\sum_{i=0}^{m-1}\langle x_i^*, x_i - x_{i+1}\rangle + \langle x_m^*, x_m - x_0\rangle \geqslant -\sum_{i=0}^{m}\alpha_i.$$

A cyclically monotone family $(T_\alpha)_{\alpha\in A}$ is called maximal if for any cyclically monotone family $(S_\alpha)_{\alpha\in A}$ satisfying $T_\alpha(x)\subset S_\alpha(x)$ for any $x\in X$ and $\alpha\in A$, one has $T_\alpha(x)=S_\alpha(x)$ for all $x\in X$ and $\alpha\in A$.

It is easily seen that the family $(\partial_\varepsilon f)_{\varepsilon\geqslant 0}$, where f is a proper (lsc) convex function on X, is a cyclically monotone family.

We claim that maximal cyclically monotone families $\{T_\varepsilon : X\to 2^{X^*}, \varepsilon\geqslant 0\}$ are exactly of the form $(\partial_\varepsilon f)_{\varepsilon\geqslant 0}$ for some $f\in\Gamma_0(X)$, X and X^* being endowed respectively with $\sigma(X,X^*)$ and $\sigma(X^*,X)$ topologies. This statement extends Rockafellar's characterization of maximal cyclically monotone operators (see [8], p. 238) to families of operators. In the particular case where X is a normed vector space, the following theorem can be deduced from Theorem 2 (part (ii) and (iii)) of Verona-Verona [10]. We give a simple proof of it in the general framework of real Hausdorff topological vector spaces using Theorem 1.2. Equality between operators will be understood in the graph sense.

THEOREM 2.1. *Let X be a real locally convex Hausdorff topological vector space, X^* its topological dual and $\{T_\varepsilon : X\to 2^{X^*}, \varepsilon\geqslant 0\}$ a cyclically monotone family of operators. Then, the family $(T_\varepsilon)_{\varepsilon\geqslant 0}$ is maximal cyclically monotone if and only if there exists a function $h\in\Gamma_0(X)$ such that $T_\varepsilon=\partial_\varepsilon h$ for all $\varepsilon\geqslant 0$.*
Moreover, the function h is unique up to an additive constant.

The proof involves the next lemma.

LEMMA 2.2. *Suppose that X and X^* are as in the above theorem and let $\{T_\varepsilon : X\to 2^{X^*}, \varepsilon\geqslant 0\}$ be any family of operators. The following are equivalent:*

(a) *$(T_\varepsilon)_{\varepsilon\geqslant 0}$ is cyclically monotone;*
(b) *there exists a function $h\in\Gamma_0(X)$ satisfying $T_\varepsilon(x)\subset\partial_\varepsilon h(x)$ for all $x\in X$ and $\varepsilon\geqslant 0$.*

Proof. The implication (b) $\Rightarrow$ (a) is immediate because of the cyclic monotonicity of the family $(\partial_\varepsilon f)_{\varepsilon\geqslant 0}$ for any proper convex function f. So, we prove (a) $\Rightarrow$ (b). Suppose that $(T_\varepsilon)_{\varepsilon\geqslant 0}$ is cyclically monotone. If

$$\bigcup\{T_\varepsilon(x), \varepsilon\geqslant 0, x\in X\}=\emptyset,$$

then any constant function $h\equiv r\in\mathbb{R}$ fits. Otherwise, choose $\varepsilon_0\geqslant 0$ and $(x_0,x_0^*)\in T_{\varepsilon_0}$. Following Rockafellar ([8], p. 238), for any $u\in X$ define

$$h(u) := \sup\left\{\sum_{i=0}^{m-1}\langle x_i^*, x_{i+1} - x_i\rangle + \langle x_m^*, u - x_m\rangle - \sum_{i=0}^{m}\varepsilon_i : \varepsilon_i \geqslant 0, (x_i, x_i^*) \in T_{\varepsilon_i}, \quad i = 1, \ldots, m, \quad m \in \mathbb{N}\right\}.$$

It is clear that $h(X) \subset \mathbb{R} \cup \{+\infty\}$ and that h is convex and $\mathrm{l_{sc}}$ on X . Further, according to the cyclic monotonicity of $(T_\varepsilon)_{\varepsilon \geqslant 0}$ one has $h(x_0) \leqslant 0$ which implies that h is proper. Fix any $\varepsilon \geqslant 0$ and $x \in X$.

If $T_\varepsilon(x) = \emptyset$, there is nothing to prove, $T_\varepsilon(x) \subset \partial_\varepsilon h(x)$. So assume the contrary and choose any $x^* \in T_\varepsilon(x)$. For each fixed $u \in X$, by definition of $h(u)$ (putting $x_{m+1} := x, x_{m+1}^* := x^*$ and $\varepsilon_{m+1} := \varepsilon$ for arbitrary $m \in \mathbb{N}$) one gets

$$h(u) \geqslant h(x) + \langle x^*, u - x\rangle - \varepsilon.$$

The above inequality being true for all $u \in X$, we conclude that $x^* \in \partial_\varepsilon h(x)$ hence $T_\varepsilon(x) \subset \partial_\varepsilon h(x)$. Thus, for such a function h, conclusion (b) holds. □

Proof of Theorem 2.1. If the family $(T_\varepsilon)_{\varepsilon \geqslant 0}$ is maximal cyclically monotone, then the conclusion follows from Lemma 2.2, where equality holds because of the maximality of the family $(T_\varepsilon)_{\varepsilon \geqslant 0}$.

To show the reverse implication, it suffices to prove that, given $f \in \Gamma_0(X)$, the family $(\partial_\varepsilon f)_{\varepsilon \geqslant 0}$ is maximal cyclically monotone. One already knows that the latter is cyclically monotone. Let us establish the maximality. Consider any cyclically monotone family $\{S_\varepsilon : X \to 2^{X^*}, \varepsilon \geqslant 0\}$, satisfying

$$\partial_\varepsilon f(x) \subset S_\varepsilon(x) \quad \text{for all } \varepsilon \geqslant 0 \text{ and } x \in X. \tag{2.1}$$

We have to prove that equality always holds in (2.1). Apply Lemma 2.2 to the family $(S_\varepsilon)_{\varepsilon \geqslant 0}$ and find some proper $\mathrm{l_{sc}}$ convex function g such that $S_\varepsilon \subset \partial_\varepsilon g$ for all $\varepsilon \geqslant 0$. By virtue of (2.1) one gets,

$$\partial_\varepsilon f(x) \subset \partial_\varepsilon g(x), \quad \forall \varepsilon \geqslant 0,\ \forall x \in X.$$

Then Theorem 1.2 yields f = g + constant on X so that

$$\partial_\varepsilon f(x) \subset S_\varepsilon(x) \subset \partial_\varepsilon g(x) = \partial_\varepsilon f(x), \quad \varepsilon \geqslant 0,\ x \in X.$$

As a result, $\partial_\varepsilon f = S_\varepsilon$ for any $\varepsilon \geqslant 0$, that ensures the maximality of $(\partial_\varepsilon f)_{\varepsilon \geqslant 0}$. The uniqueness is a direct consequence of the integration Theorem 1.2. □

References

1. Brøndsted, A. and Rockafellar, R.T. (1965), On the subdifferentiability of convex functions, *Proceedings of American Mathematical Society* 16, 605–611.

2. Combari, C., Marcellin, S. and Thibault, L. (2003), On the graph convergence of ε-Fenchel subdifferentials of convex functions. *Journal of Nonlinear and Convex Analysis* 4(2), 309–324.
3. Hiriart-Urruty, J.-B. (1982), *ε-Subdifferential Calculus*, *Convex analysis and optimization*, pp. 43–92, Research Notes in Mathematics, 57. Pitman, Boston, Mass.-London.
4. Hiriart-Urruty, J.-B. and Phelps, R.R. (1993), Subdifferential calculus using ε-subdifferentials, *Journal of Functional Analysis* 118, 154–166.
5. Hiriart-Urruty, J.-B., Moussaoui, M., Seeger, A. and Volle, M. (1995), Subdifferential calculus without qualification conditions, using approximate subdifferentials: a survey, *Nonlinear Analysis* 24, 1727–1754.
6. Martinez-Legaz, J.-E. and Théra, M. (1996), ε-Subdifferentials in terms of subdifferentials, *Set-Valued Analysis* 4, 327–332.
7. Moreau, J.-J. (1966–1967), Fonctionnelles convexes, Séminaire sur les équations aux dérivées partielles, Collège de France.
8. Rockafellar, R.T. (1970), *Convex analysis*, Princeton, New Jersey, Princeton University Press.
9. Thibault, L. (2000), Limiting convex subdifferential calculus with applications to integration and maximal monotonicity of subdifferential, CMS Conference. *Proceedings of American Mathematical Society*, Providence, RI, 27, 279–289.
10. Verona, A. and Verona, M.E. (1993), Remarks on subgradients and ε-subgradients, *Set-Valued Analysis* 1, 261–272.